Indian Statistical Institute
M. Math 2nd year
Academic year 2022-2023
Endsem Examination
Course: Special Topics in Geometry: Harmonic maps
02 - 05 - 2023
3 hours

- *Answer as many questions as you can.*
- *You may use results proved in class, but make sure to state them clearly.*
- *Maximum marks is 60.*

1. Let $M_n(\mathbb{R})$ denote the vector space of $n \times n$ real matrices, and let $Sym_n(\mathbb{R})$ denote the vector subspace of symmetric $n \times n$ matrices, where both vector spaces are equipped with the operator norm. Let $O_n(\mathbb{R})$ denote the set of $n \times n$ orthogonal matrices.

   Let $f : M_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ be defined by $f(A) := A^T A$ for $A \in M_n(\mathbb{R})$.

   (a) Show that f is differentiable and compute the derivative $Df_A : M_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ for any $A \in M_n(\mathbb{R})$.

   (b) Show that if $A \in O_n(\mathbb{R})$ then Df_A has full rank.

   (5 + 5 = 10 marks)

2. Let $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear function, and let $F : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $F(x) := A(x, x),\ x \in \mathbb{R}^n$. Show that F is twice differentiable and $D^2F_x = 2A$ for all $x \in \mathbb{R}^n$. (10 marks)

3. Let M be a smooth manifold, let X be a complete vector field on M and let $(\phi_t : M \to M)_{t\in\mathbb{R}}$ be the flow of X. Let ω be a k-form on M. Show that the Lie derivative of ω with respect to X satisfies $\mathcal{L}_X\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}$ if and only if $\phi_t^*\omega = e^{\lambda t}\omega$ for all $t \in \mathbb{R}$. (8 marks)

4. Let M be a compact Riemannian manifold without boundary.

(a) Show that

$$\text{div}(\phi X) = < \nabla \phi, X > + \phi \ \text{div} X$$

for any smooth function ϕ on M and any smooth vector field X on M.

(b) Let ϕ be a smooth subharmonic function on M, i.e. $\Delta \phi \geq 0$ on M. Show that ϕ is constant. (8+4 = 12 marks)

5. Let M, N be Riemannian manifolds. Let $f : M \to N$ be a smooth map. Show that f is harmonic if and only if, for any $p \in M$, if u is a smooth convex function in a neighbourhood of $f(p)$ in N, then $u \circ f$ is subharmonic in a neighbourhood of p. (12 marks)

6. Let M, N be compact Riemannian manifolds without boundary, and suppose the sectional curvature of N is nonpositive. Let $(f_t : M \to N)_{t \in [0,1]}$ be a smooth homotopy between maps f_0 and f_1, and suppose the homotopy is a *geodesic homotopy*, i.e. for any $p \in M$, the curve $t \mapsto f_t(p)$ is a geodesic in N.

(a) Show that the energy $E(f_t)$ of the maps $f_t : M \to N$ satisfies $\frac{d^2}{dt^2} E(f_t) \geq 0$ for all $t \in [0, 1]$.

(b) Suppose the map f_0 has minimum energy in its homotopy class. Show that if f_1 is harmonic, then $E(f_t) = E(f_0)$ for all $t \in [0, 1]$, and the maps f_t are harmonic maps for all $t \in [0, 1]$. (10+6 = 16 marks)